Faculty of Science, Technology, Engineering and Mathematics
M208 Pure mathematics

M208

TMA 04 2019J

Covers Book D **Cut-off date 13 February 2020**

You can submit your TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on the TMA, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' tab of the M208 website.

In the wording of the questions:

- *write down*, *list* or *state* means 'write down without justification' (unless otherwise stated)
- *find*, *determine*, *calculate*, *derive*, *evaluate* or *solve* means 'show all your working'
- *prove*, *show*, *deduce* or *verify* means 'justify each step'
- *sketch* means 'sketch without justification' and *describe* means 'describe without justification' (both unless otherwise stated).

In particular, if you use a definition, result or theorem to go from one line to the next, then you must state clearly which fact you are using – for example, you could quote the relevant unit and page, or give a Handbook reference. Remember that when you use a theorem, you must demonstrate that all the conditions of the theorem are satisfied.

The number of marks assigned to each part of a question is given in the right-hand margin, to give you a rough indication of the amount of time that you should spend on each part.

Your work should be written in a good mathematical style, as demonstrated by the exercise and worked exercise solutions in the study texts. You should explain your solutions carefully, using appropriate notation and terminology, defining any symbols that you introduce, and writing in proper sentences. Five marks (referred to as good mathematical communication, or GMC, marks) on this TMA are allocated for how well you do this.

Your score out of 5 for GMC will be recorded against Question 6. (You do not have to submit any work for this particular question.)

You should read the information on the front page of this booklet before you start working on the questions.

Question 1 (Unit D1) – 25 marks

(a) Solve the inequality

$$\frac{x+1}{x+2} \geq \frac{x-4}{2x-3}.$$ [7]

(b) Prove that $n! \leq n^n$, for $n \geq 1$. [6]

(c) Use the Binomial Theorem to prove that

$$\left(1+\frac{2}{3n}\right)^n \geq \frac{17}{9} - \frac{2}{9n}, \quad \text{for } n \geq 1.$$ [6]

(d) Determine the least upper bound of the set

$$E = \left\{2 - \frac{3}{n^3} : n = 1, 2, \ldots\right\}.$$ [6]

Question 2 (Unit D2) – 15 marks

Determine whether or not each of the following sequences (a_n) converges. You may use the basic null sequences listed in Theorem D7 from Unit D2, but you should state clearly which results or rules you use. Find the limit of each convergent sequence.

(a) $a_n = \dfrac{5-2n^2+3n^4}{n^4+8n^3-5}, \quad n = 1, 2, \ldots$ [4]

(b) $a_n = \dfrac{n^3-7}{n^2+10n-9}, \quad n = 1, 2, \ldots$ [5]

(c) $a_n = \dfrac{n^3+(-1)^n(n!)}{2n^2+4^n+3(n!)}, \quad n = 1, 2, \ldots$ [6]

Question 3 (Unit D3) – 19 marks

This question concerns the following exercise.

Exercise

Determine whether or not each of the following series converges or diverges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2+1}{2n^2+3}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{3n^4-2}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 5^n}{(n+3)(n!)}$

(a) Explain why the following solution is incorrect, identifying at least one error in each of the three parts. (There may be more errors, but you are required to identify only one in each part.) [7]

Solution (incorrect!)

(i) Let $a_n = \dfrac{4n^2+1}{2n^2+3}$. Dividing the numerator and denominator by the dominant term n^2, we have by the Combination Rules and the list of basic null sequences that

$$a_n = \frac{4n^2+1}{2n^2+3} = \frac{4+1/n^2}{2+3/n^2} \to \frac{4+0}{2+0} = 2 \quad \text{as } n \to \infty.$$

So the given series converges.

(ii) Let $a_n = \dfrac{3n^2+2}{3n^4-2}$. For $n = 1, 2, \ldots$ we have that

$$0 < \frac{3n^2+2}{3n^4-2} < \frac{3n^2}{3n^4} = \frac{1}{n^2}.$$

Since a_n is positive for $n = 1, 2, \ldots$ and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (it is a basic convergent series, by Theorem D33 from Unit D3), it follows by the Comparison Test that the given series converges.

(iii) Let $a_n = \dfrac{n^2 5^n}{(n+3)(n!)}$ and let $b_n = \dfrac{5^n}{n!}$. Then both a_n and b_n are positive for $n = 1, 2, \ldots$, and

$$\frac{a_n}{b_n} = \frac{n^2 5^n}{(n+3)(n!)} \times \frac{n!}{5^n} = \frac{n^2}{n+3}.$$

Now

$$\frac{b_n}{a_n} = \frac{n+3}{n^2} = \frac{1/n+3/n^2}{1} \to 0 \quad \text{as } n \to \infty,$$

so by the Reciprocal Rule $a_n/b_n \to \infty \neq 0$ as $n \to \infty$. Since the series $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent (it is a basic convergent series, by Theorem D33 from Unit D3), it follows by the Limit Comparison Test that the given series converges.

(b) Write out a correct solution to the exercise, justifying your answer carefully. You may use the basic series listed in Theorem D33 from Unit D3, but you should state clearly which results or rules you use. You may need to apply convergence tests different from those used above. [12]

Question 4 (Unit D3) – 11 marks

Use Strategy D13 from Unit D3 to determine which of the following series are convergent. You may use the basic series listed in Theorem D33 from Unit D3, but you should state clearly which results or rules you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{6n^2-5}$ [5]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5n+2\sin n}{4n^3+3}$ [6]

Question 5 (Unit D4) – 25 marks

(a) Determine which of the following functions are continuous at 1. You may use the basic continuous functions listed in Theorem D51 from Unit D4, but you should state clearly which results or rules you use.

(i) $f(x) = \begin{cases} 3x^3 - x^2, & x < 1, \\ 1 - \cos(\pi x), & x \geq 1. \end{cases}$ [6]

(ii) $f(x) = \begin{cases} (x-1)^4 \sin\left(\dfrac{1}{x-1}\right), & x \neq 1, \\ 0, & x = 1. \end{cases}$ [7]

(iii) $f(x) = \begin{cases} 4x^3 - 1, & x \leq 1, \\ x^2 + 3, & x > 1. \end{cases}$ [4]

(b) The polynomial p is given by

$$p(x) = 2x^3 + 4x^2 - 1.$$

(i) Determine a positive integer N such that p has no zeros outside the interval $(-N, N)$, and exactly three zeros in $(-N, N)$. [6]

(ii) Determine an open interval of length $\frac{1}{2}$ in $\mathbb{R}$ that contains exactly one zero of p. [2]

Question 6 (Book D) – 5 marks

Five marks on this assignment are allocated for good mathematical communication in your answers to Questions 1 to 5.

You do not have to submit any extra work for Question 6, but you should check through your assignment carefully, making sure that you have explained your reasoning clearly, used notation correctly and written in proper sentences. [5]